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The historical bases of the Rayleigh and Ritz methods

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Abstract

Rayleigh's classical book *Theory of Sound* was first published in 1877. In it are many examples of calculating fundamental natural frequencies of free vibration of continuum systems (strings, bars, beams, membranes, plates) by assuming the mode shape, and setting the maximum values of potential and kinetic energy in a cycle of motion equal to each other. This procedure is well known as "Rayleigh's Method." In 1908, Ritz laid out his famous method for determining frequencies and mode shapes, choosing multiple admissible displacement functions, and minimizing a functional involving both potential and kinetic energies. He then demonstrated it in detail in 1909 for the completely free square plate. In 1911, Rayleigh wrote a paper congratulating Ritz on his work, but stating that he himself had used Ritz's method in many places in his book and in another publication. Subsequently, hundreds of research articles and many books have appeared which use the method, some calling it the "Ritz method" and others the "Rayleigh–Ritz method." The present article examines the method in detail, as Ritz presented it, and as Rayleigh claimed to have used it. It concludes that, although Rayleigh did solve a few problems which involved minimization of a frequency, these solutions were not by the straightforward, direct method presented by Ritz and used subsequently by others. Therefore, Rayleigh's name should not be attached to the method.

1. Introduction

In 1877, the first edition of *Theory of Sound* by Lord Rayleigh [1] was published. (See the excellent historical introduction by Lindsay, in the 1945 reprinting, for an excellent biographical

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sketch of Rayleigh and his work.) Volume I is devoted to vibration concepts, and the underlying mathematics, and also contains six chapters dealing with vibrations of strings, bars, beams, membranes, plates and shells. Volume II [2], published one year later, addresses problems in acoustics.

Most of the numerous problems dealt with in Volume I begin with the governing differential equation of motion, and are solved by classical methods, applying boundary conditions to obtain free vibration frequencies and mode shapes. But Rayleigh was also interested in the potential (V) and kinetic (T) energies of the system and, in some cases, attacked the problems from this perspective. In particular, in many cases, he assumed a mode shape, and calculated the corresponding free vibration frequency by equating V and T during a vibration cycle. This has generally become known as the Rayleigh method of solution. Its accuracy depends upon how closely the assumed mode shape fits the correct (exact) one.

In 1908 and 1909, Walter Ritz [3,4] published two papers that thoroughly demonstrated a straightforward procedure for solving boundary value and eigenvalue problems numerically, to any degree of exactitude desired, also using energy functionals. For the free vibration eigenvalue problem, one assumes a displacement function in terms of a series of admissible displacement functions (that is, ones satisfying at least the geometric boundary conditions of the problem) having undetermined coefficients, and then minimizes an energy functional involving V and T to determine frequencies and mode shapes. The first paper [3] was extensive (61 pages), and laid out the method and its underlying concepts, discussed convergence, and applied it to some problems. The second one [4] used the method to present novel results for the vibrations of a completely free square plate. Tragically, Ritz died of consumption soon afterwards (cf. Ref. [5]). His complete scientific works were collected together and published posthumously [6].

Although the Rayleigh method is used frequently, the Ritz method has found tremendous usage during the past three decades in obtaining accurate frequencies and mode shapes for the vibrations of continuous systems, especially for problems not amenable to exact solution of the differential equations. This is especially because of the increasing capability of digital computers to set up and solve the frequency determinants arising with the method. Even before that, the writer found 15 publications that used the Ritz method to solve classical rectangular plate vibration problems prior to 1966. These are described in Chapter 4 of his plate vibration monograph [7].

In going through the 15 papers (as well as others, used elsewhere in the monograph for other shapes of plates, or non-classical ones), the author became aware that many researchers had also attached Rayleigh's name to the Ritz method, calling it "the Rayleigh–Ritz method." At that time he regarded this as simply a way of amalgamating the two methods, because the Rayleigh method may be regarded as a special case of the Ritz method when only a single admissible function is used to describe the vibration mode. But this is misleading, because then one would not bother to write the Ritz minimizing equation—the Rayleigh procedure is more direct.

As time went on, the writer heard comments more than once that Rayleigh had used the Ritz method, and had written about it. But those who spoke could not cite references, saying that "they must be in *Theory of Sound*." The writer has looked through these volumes many times during the past 40 years, and never found anything closely resembling the Ritz method. However, recently he was made aware of some additional published papers by Rayleigh, notably one [8] he published two years after Ritz's second paper, wherein he complained that Ritz had not recognized his own, similar work.

Perhaps because Rayleigh himself claimed to deserve sharing credit for the Ritz method, many subsequent researchers attached his name to it, without looking for verification. Therefore, the primary purpose of this paper is to investigate carefully the historical basis for Rayleigh's complaint and claim, in an attempt to determine their validity. To do so requires also looking carefully at exactly what Ritz did. A secondary purpose is to better acquaint many researchers who currently use the Ritz's method with its sources, especially since they are in German [3,4].

2. The method of Rayleigh

Consider first the most simple example of a one degree of freedom spring–mass system vibrating freely, with a linear spring. The classical differential equation of motion is

$$m\ddot{x} + kx = 0. \tag{1}$$

If the initial conditions of the motion are $x(0) = A$, $\dot{x}(0) = 0$, then the well-known solution to Eq. (1) is

$$x = A \cos \omega t, \tag{2}$$

where $\omega^2 = k/m$ is the natural frequency.

Also well known is the energy approach to the same problem. During a displacement x, the potential energy (V) stored in the spring at any instant of time is $kx^2/2$, while the kinetic energy of the system (T) is in the mass (assuming a massless spring), so that $T = m\dot{x}^2/2$. With no damping, or other externally applied forces, the system is conservative; i.e.,

$$T + V = \text{constant}. \tag{3}$$

During a cycle of motion, described by Eq. (2), for example, the total energy interchanges between T and V. At any instant,

$$V = \tfrac{1}{2}kA^2 \cos^2 \omega t, \quad T = \tfrac{1}{2}mA^2\omega^2 \sin^2 \omega t. \tag{4}$$

The maximum potential energy ($V_{\max}$) occurs when $\cos^2 \omega t = 1$. But then the mass has no velocity, so $T = 0$. The maximum kinetic energy ($T_{\max}$) develops as the mass passes through the equilibrium position with maximum velocity, so that $\sin^2 \omega t = 1$. At this instant $V = 0$. Because the total energy of the system remains constant, then

$$V_{\max} = T_{\max}. \tag{5}$$

Thus, substituting $V_{\max} = kA^2/2$ and $T_{\max} = mA\omega^2/2$ into Eq. (5), one again arrives at $\omega^2 = k/m$.

The above is well known to anyone who works in the field of vibrations. It is laid out only to clarify some concepts, and to establish the notation clearly.

Consider next the multiple degrees of freedom discrete system depicted in Fig. 1. For simplicity, three equal masses (m) connected by four springs having equal stiffnesses (k) are shown. From three free body diagrams and Newton's laws one may write the governing three equations of motion in terms of the displacements, $x_i (i = 1, 2, 3)$. Assuming normal modes ($x_i = A_i \cos \omega t$) leads to the third-order determinant for the eigenvalues (nondimensional frequencies), which are $\lambda_1 = 2 - \sqrt{2}$, $\lambda_2 = 2$, $\lambda_3 = 2 + \sqrt{2}$ ($\lambda_i = m\omega_i^2/k$). Eigenvectors and mode shapes are obtained by

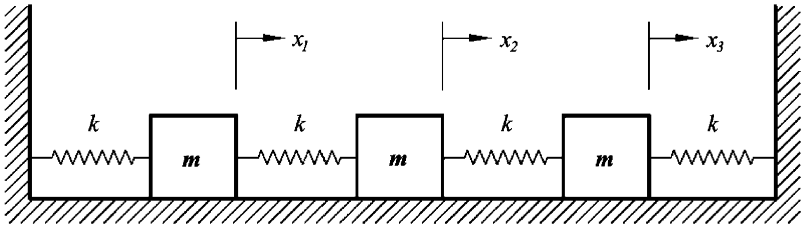

Fig. 1. Three degrees of freedom, discrete system.

back-substitution in the usual manner, yielding $A_2/A_1 = \sqrt{2}$, $A_3/A_1 = 1$ for the first mode, for example.

The same problem could be solved by an energy approach. The potential energy in the springs at time t is

$$V = \frac{k}{2}x_1^2 + \frac{k}{2}(x_2 - x_1)^2 + \frac{k}{2}(x_3 - x_2)^2 + \frac{k}{2}x_3^2. \tag{6}$$

The kinetic energy of the system at a typical time is

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}m\dot{x}_3^2. \tag{7}$$

Using these with Lagrange's equation of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \quad (i = 1, 2, 3), \tag{8}$$

where $L \equiv T - V$, yields the same three equations of motion that one obtains from the free body diagrams and Newton's laws. The solution of the eigenvalue problem to find all three natural frequencies and mode shapes is then the same as before.

The essence of the Rayleigh method for the foregoing problem is to *assume* a mode shape. For example, as an estimate of the fundamental mode shape, one could assume that $A_2/A_1 = 2$ and $A_3/A_1 = 1$ for the amplitudes in Eq. (2). This assumes that this mode shape is symmetric ($A_3 = A_1$) for these identical masses and springs, and that the amplitude of the middle mass will be significantly larger than that of the outer ones ($A_2/A_1 = A_2/A_3 = 2$). Using these amplitudes in Eq. (2), and substituting the latter into Eqs. (6) and (7), one finds that $V_{\max} = 2kA_1^2$ and $T_{\max} = 3m\omega^2 A_1$. Whence, setting $V_{\max} = T_{\max}$ results in $\lambda = 2/3$. Thus, the fundamental frequency obtained from this approximate solution is $\omega_1 = 0.8165\sqrt{k/m}$, which is 6.7% higher than the value of $0.7654\sqrt{k/m}$ from the previous exact solution.

The exact amplitude ratios for the fundamental mode were found to be $A_2/A_1 = A_2/A_3 = \sqrt{2} = 1.414$, compared with the assumed value of 2 used in the Rayleigh method. If the exact amplitude ratios had been assumed, the Rayleigh method would yield the exact corresponding frequency.

As an example of a *continuous* system, consider the longitudinal (x-direction) vibrations of the homogeneous, uniform bar of length l shown in Fig. 2. The well-known differential equation for it is the classical one-dimensional wave equation

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{9}$$

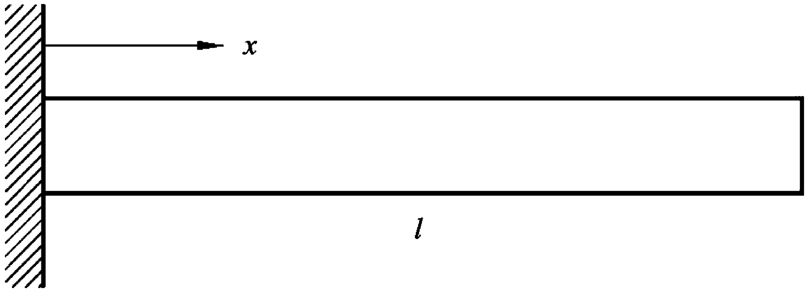

Fig. 2. Uniform bar, continuous system.

where $u = u(x, t)$ is the longitudinal displacement, E is Young's modulus, and ρ is mass density (per unit volume). Separating variables, solving the resulting ordinary differential equations, and applying the fixed-free end conditions shown in Fig. 1, one arrives at

$$u_m(x, t) = C_m \sin \alpha_m x \cos \omega_m t, \tag{10}$$

for the mth free vibration mode, if the bar is released from rest in the $\sin \alpha_m x$ mode shape with amplitude C_m. From the fixed-free end conditions one finds

$$\alpha_m = \frac{(2m-1)\pi}{2l} \quad (m = 1, 2, \ldots, \infty), \tag{11}$$

and the nondimensional frequencies

$$\omega_m l \sqrt{\frac{\rho}{g}} = \frac{(2m-1)\pi}{2} \quad (m = 1, 2, \ldots, \infty). \tag{12}$$

Using the Rayleigh method instead to obtain an approximation to the fundamental frequencies, the potential energy (which is strain energy here) is needed,

$$V = \frac{1}{2}\int_0^l AE\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x, \tag{13}$$

as well as the kinetic energy

$$T = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x. \tag{14}$$

Here A is the cross-sectional area of the rod. If $A \neq A(x)$, that is, for the uniform bar we are considering, then the frequencies and mode shape are found to be independent of A.

If the displacement (u) is assumed to vary *linearly* along the length of the bar in its fundamental free vibration mode shape, then

$$u(x, t) = Cx \cos \omega t, \tag{15}$$

where Cl is the amplitude of the motion at the free end. Substituting Eq. (15) into Eqs. (13) and (14) one obtains

$$V_{\max} = \left(\frac{AEl}{2}\right)C^2, \quad T_{\max} = \left(\frac{\omega^2 \rho A l^3}{6}\right)C^2, \tag{16}$$

whence $\omega l\sqrt{\rho/E} = \sqrt{3} = 1.732$, which is 10.3% higher than the exact fundamental frequency of $\pi/2 = 1.571$ given in Eq. (12). Of course, the straight line mode shape assumed in Eq. (15) is considerably different from the one-quarter sine wave (10) of the exact solution. If the exact shape were assumed, the exact corresponding frequency would be generated by the Rayleigh method.

As can be seen in the foregoing two examples, the accuracy of the Rayleigh method depends entirely upon how well one estimates the shape of the free vibration mode desired. It may be used to obtain approximate frequencies for higher modes, in addition to the fundamental. And, as the examples showed, unless one assumes the exact mode shape, one obtains a frequency that is too high.

The method is particularly useful if one does not have reliable structural mechanics computer programs (e.g., finite elements), or if one wants to make a quick check of the accuracy of a program. For the first reason, the method was used a great deal 30 and more years ago. An excellent example of this is the classical paper by Warburton [9] in which he used the Rayleigh method to derive formulas for the frequencies of rectangular plates having all 21 possible combinations of clamped, simply supported or free edges. He used mode shapes, which are the products of vibrating beam eigenfunctions (satisfying the clamped, simply supported or free boundary conditions). These single-term mode shape representations were found to be quite accurate (<1% error in the frequencies), except when free edges are involved [10].

In a few places in his first classic book, Rayleigh [1] used the method described above, which now bears his name. A good example is to be found in Section 89 (pp. 112–113) where he solved the well-known problem of the vibrating string by assuming

$$w(x,t) = [1 - (2x/l)^n]\cos\omega t \tag{17}$$

for the fundamental mode, choosing $x = 0$ to be the middle of the string, instead of the well-known $\cos(\pi x/2l)$ exact one, when both ends of a string of length $2l$ are fixed. Interestingly, he used Eq. (17) to derive and present T and V in terms of $\sin\omega^2 t$ and $\cos\omega^2 t$, respectively, from which he concluded that

$$\omega^2 = \frac{2(n+1)(2n+1)}{2n-1}\frac{T_1}{\rho l^2}, \tag{18}$$

where T_1 is the tension in the string and ρ is its mass density, but he does not mention that V_{max} must equal T_{max} for result (18) to follow. This point could be assumed to be self-evident. But the present writer could find the explicit statement of Eq. (5) anywhere in Rayleigh's book [1].

Rayleigh did make a clear statement in Section 89 (as he does elsewhere) that the approximate frequency given by Eq. (18) is higher then the exact value, because in assuming Eq. (17) for the mode shape, the string is constrained to vibrate otherwise than it would naturally, and that adding a constraint to a system increases its frequency. In detail, Rayleigh showed that choosing $n = 1$ yields $\lambda \equiv \omega^2\rho l^2/T_1 = 12$, instead of the exact value, $\pi^2 = 9.8696$, using $n = 2$ gives a better result, $\lambda = 10$, and that a minimum λ (the best-possible result from Eq. (18)) is 9.8990 when $n = (\sqrt{6}+1)/2 = 1.72474$, although he does not mention how he determined that minimum. Again, perhaps this was too obvious.

The concept of assuming a displacement shape (i.e., function) and using this as an approximate solution is applicable also to solve *static* problems, although the writer does not believe that Rayleigh's name should be attached to this. In Rayleigh's published works, the idea is found only in connection with free vibration problems. Nevertheless, Temple and Bickley [11] in their short monograph devoted to the concept, attach Rayleigh's name to eigenvalue problems of static elastic stability (determining critical loads) as well, but not to simple static deflection situations. In a static problem, one sets the work done by the loads (that is, the negative of the load potential) equal to the strain energy of deformation (i.e., $W = V$). In ordinary problems this procedure yields the amplitude of the displacement (from which one may calculate stresses to lesser a accuracy). In elastic stability, the work done and strain energy are due to infinitesimal displacements away from a static equilibrium position.

3. The method of Ritz

The first paper that Ritz wrote to describe his method for solving boundary value and eigenvalue problems was published in 1908 [3] ("On a New Method for the Solution of Certain Variational Problems of Mathematical Physics"). The paper not only presented the method, but demonstrated and described it for several static equilibrium and free vibration problems. In the introduction, he explained that methods of solving boundary value problems at that time were typically impractical, and this was his primary motivation for developing the new method.

As a prologue to the introduction, Ritz described the one-dimensional variational problem of minimizing a functional,

$$J = \int_a^b f(x, w, w', w'', \ldots)\,\mathrm{d}x, \tag{19}$$

by assuming functions $w_n(x)$ in the form of series

$$w_n(x) = \sum_{i=1}^{n} a_i \psi_i(x), \tag{20}$$

where the ψ_i may be, for example, algebraic polynomials or trigonometric functions (essential boundary conditions are not yet mentioned), and the a_i are arbitrary coefficients. The functional J is minimized by simply taking the partial derivatives

$$\frac{\partial J}{\partial a_i} = 0 \quad (i = 1, 2, \ldots, n). \tag{21}$$

As he said, for typical (in his day) linear problems, J is a quadratic form, and the resulting equations (21) are linear in the a_i. For static equilibrium (boundary value) problems, the equations are nonhomogeneous, and one solves directly for the a_i. For free vibration (eigenvalue) problems, x does not appear explicitly as an independent variable in Eq. (19), Eqs. (21) are homogeneous, and an eigenvalue determinant arises from the equations.

It should be remarked here that the integrand of Eq. (19) is standard for one-dimensional structural mechanics problems. For example, $f(x, w, w')$ would accommodate the longitudinal elastic stretching of a bar (uniform or nonuniform), or its twisting (not considering warping constraint), or the transverse displacement of a stretched string. Similarly, $f(x, w, w', w'')$ could deal with the transverse displacements of a classic (Euler–Bernoulli) beam.

The main part (pp. 6–44) of Ritz's paper [3] deals with the application of his method to the classic Kirchhoff plate problem ("Deformation of an Elastic Plate Clamped on its Boundary, under the Influence of a Given Normal Pressure"). Although its title appears to limit the analysis to the static equilibrium problem, the free vibration case is also discussed at the end of this part.

The governing biharmonic differential equation for the problem was exhibited:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = f(x, y), \tag{22}$$

along with the clamped boundary conditions

$$w = \frac{\partial w}{\partial n} = 0, \tag{23}$$

and the corresponding energy functional

$$J = \frac{1}{2}\int_A\int [(\nabla^2 w)^2 - f(x, y)w]\,\mathrm{d}A. \tag{24}$$

In these forms, Ritz was obviously now setting the plate flexural rigidity, $D \equiv Eh^3/12(1 - \nu^2)$, equal to unity to avoid dragging along this constant through the calculations. He also pointed out correctly that the energy functional (strain energy plus transverse load potential) is simplified to the form of Eq. (24) if w satisfies Eqs. (23). Thus, the first problem that Ritz actually exhibited and analyzed in detail was more complicated than the one-dimensional one described in his Introduction.

He then continued by discussing various mathematical aspects involved when applying the new method to the clamped square plate problem; for example: (1) a proof (p. 8) that exact minimization of the energy functional (24) yields the differential equation (22); (2) an approximate minimization of J yields an upper bound for it (p. 15); (3) a proof that the limiting solution of $w_n(x, y)$ converges uniformly to the exact solution of the differential equation if proper two-dimensional functions $\psi(x, y)$ are used in the approximate solution (20), and n is increased as needed.

Considerable space (pp. 25–40) was next devoted to how one may choose admissible functions $\psi_{ij}(x, y)$ so as to satisfy the clamped boundary conditions in using the approximate solution

$$w_{mn}(x, y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\psi_{ij}(x, y), \tag{25}$$

in Ritz's method. First, the general case of a polygonal plate having clamped edges was discussed, but then he returned to the clamped square plate, suggesting the following types of ψ_{ij}: (1) Fourier sine series: $\sin(i\pi x/a)\sin(j\pi y/a)$; (2) ordinary polynomials: $x^i y^j$; (3) Legendre (orthogonal) polynomials: $P_i(x)P_j(y)$; provided that each term is multiplied by the equation of the boundary squared (F), so that each satisfies the boundary conditions; that is, multiplied by

$$F = (x - a)^2(x - a')^2(y - b)^2(y - b')^2, \tag{26}$$

where a, a', b, b' are the boundary locations in the xy-coordinate system. Using the ordinary polynomials of Eq. (25), premultiplied by F, is the form used by many analysts today, although in more generality, the exponents in Eq. (26) are taken to be zero, one or two, depending upon whether the particular boundary is free, simply supported, or clamped, respectively.

Ritz then introduced (p. 33) the concept of taking the ψ_{ij} in Eq. (25) to be the product of beam vibration eigenfunctions; i.e.,

$$\psi_{ij}(x, y) = X_i(x)Y_j(y), \tag{27a}$$

where X_i satisfies the beam boundary conditions at the end of the x interval, and similarly for Y_j. For his C–C–C–C plate, of course, Ritz used the clamped–clamped beam eigenfunctions in both directions, exhibiting them in detail, and discussing their orthogonality and the advantages of this in carrying out calculations.

The problem of the C–C–C–C square plate subjected to uniform pressure was solved in detail, beginning on p. 41. Two approximate numerical solutions were presented, one employing the first three, independent, doubly symmetric products of X_iY_j in Eq. (25)—that is, retaining only a_{11}, $a_{31}(= a_{13})$, a_{33} because of the plate and loading symmetry, and another solution using the first six ($a_{11}, a_{31} = a_{13}, a_{33}, a_{51} = a_{15}, a_{53} = a_{35}, a_{55}$). These showed the rapidity of convergence, with the additional three terms turning out to be very small.

On p. 44, Ritz discussed briefly the application of his method to plate *free vibration* problems, for which Eq. (22) is replaced by

$$\nabla^4 w = \lambda w, \tag{27b}$$

and, similarly, $f(x, y)$ is replaced by λw in the energy functional (24) to be minimized. This is an eigenvalue problem. His method would generate a characteristic determinant, the roots of which are the desired eigenvalues (nondimensional frequencies), λ.

Having dealt extensively with the clamped square plate, Ritz then turned (p. 45) to the less-complicated Dirichlet problem governed by the differential equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{28}$$

with either u or $\partial u/\partial n$ specified on the boundary. If u takes on values $Q(x, y)$ along the boundary, Ritz shows that by defining a new function $w = u - Q$, Eq. (10) is replaced by

$$\nabla^2 w = f(x, y), \tag{29}$$

where w is now zero on the boundary. The corresponding functional to be minimized is

$$J = \frac{1}{2}\int_A\int\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 - 2wf(x, y)\right]\mathrm{d}x\,\mathrm{d}y. \tag{30}$$

This corresponds to twice the potential energy in a taut membrane subjected to transverse static pressure, $f(x, y)$. Most of the rest of this section of his paper is devoted to a mathematical discussion of his minimization method when applied to the Dirichlet problem, including convergence. But no specific problems are solved.

A short section (pp. 52–57) titled "Linear Differential Equations with Variable Coefficients" is presented, which is applicable to certain one-dimensional problems, such as the longitudinal deformation of a nonhomogeneous and/or variable cross-section rod. No specific problems are solved.

The last section (pp. 57–61) goes into the details of solving the vibrating string problem. Taking the coordinate origin at the middle of the string, the boundaries ($x = \pm 1$) are fixed by choosing the transverse displacement (y) as

$$y_n = (1 - x^2)(a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n}), \tag{31}$$

which considers only the *symmetric* vibration modes, which are uncoupled from the antisymmetric modes. The energy functional to be minimized is

$$J_n = \int_{-1}^{1} [(y'_n)^2 - k_n^2 y_n^2]\, \mathrm{d}x, \tag{32}$$

where the first term in the integrand is from the potential energy, and the second from the kinetic energy. Setting up the minimizing equations $\partial J_n/\partial a_i = 0$ yields the characteristic determinant for the eigenvalues (nondimensional frequencies). The *exact* solution for the fundamental frequency is $2k_1^2 = \pi^2/2 = 4.934802200$ (Ritz said, which is correct to nine significant figures, and is one indication of the accuracy to which he worked a century ago). Retaining only a_0 and a_2 in Eq. (31), Ritz obtained $2k_1^2 = 4.93488$, a reasonably accurate upper bound. Adding a third term (a_4), and expanding the third-order determinant, he arrived at the very accurate approximation $2k_1^2 = 4.934802217$ for the fundamental frequency. He also showed that the second approximate eigenvalues were less accurate upper bounds for the second symmetric mode frequency (third mode, overall). Finally, he concluded this numerical study by comparing the approximate and exact fundamental mode shapes, as well as the location of the node points for the second symmetric mode.

We translate from the German the last page of Ritz's first article [3], which contained certain noteworthy statements: "One may conclude from this example that our method for the calculation of the fundamental frequency of a string, membrane or plate is particularly advantageous, and that the higher frequencies are with which one is involved, the greater the number of constants a_i one needs to calculate to achieve a given exactitude."

Continuing (p. 61): "The method is also useful for the investigation of Chladni vibration mode shapes. For the transversely vibrating plate, in the case of a free boundary, this problem has thus far been solved only for the circle. But for the rectangular plate clamped along its boundary the solution is yet unknown. It is easy to show that the corner points are singular, and in their neighborhoods a development in power series is not possible. Here our method, for example, with application of polynomials, also has a great advantage in being applicable, as long as the essential boundary conditions are not violated. For this application of the new method to the transverse vibrations of a flat rectangular plate I shall return to in another place."

Ritz did indeed return to this problem, and quickly. One year later, he published his second paper [4] ("Theory of Transverse Vibrations of a Square Plate with Free Edges") using the new method. Using the products of vibrating beam eigenfunctions as admissible approximation solutions in Eq. (25), the numerical solutions are somewhat more complicated than those for the completely clamped plate, which he had discussed in his first paper. This is because the strain energy of bending has a term

$$-2(1 - \nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right], \tag{33}$$

(where v is Poisson's ratio) that must be added to the energy functional integrand of Eq. (24), and does not vanish upon integration if part of the boundary is free. He may have chosen this problem because it was one that Chladni had investigated experimentally a century earlier [12–15], and others (including Rayleigh) had examined theoretically [16,17], but for which no satisfactory solution existed. We also know that no exact solution for it had been found. Nor has any been found yet, one century later.

In this second paper, Ritz presented frequencies and nodal patterns (also called "Chladni figures" then) for numerous mode shapes of the F–F–F–F square plate, using $v = 0.255$. As many as six terms were retained in Eq. (25) in order to obtain the lower frequencies reasonably accurately.

4. Rayleigh's complaint

In 1911, two years after the publication of Ritz's second paper [4], which demonstrated his solution method in detail for the free vibration problem of the completely square plate, Rayleigh published a paper [8], which first complimented Ritz on his work, but then took him to task for not acknowledging some earlier work which Rayleigh felt was very similar.

In the first paragraph, Rayleigh [8] mentioned that he had discussed the completely free square plate in Chapter 10 of his book [1] (see pp. 372–383), saying that the problem was "one of great difficulty, which for the most part resisted attack." He then went on to say that an exception is the case when Poisson's ratio (v) for the material is zero. Of course, then the two-dimensional anticlastic effects vanish, and the plate behaves as a beam. In his book, Rayleigh devotes most of the space (pp. 373–379) to showing how combinations of the 1-D beam modes could be superimposed to obtain nodal patterns similar to those found by Chladni. By this approach then, for a square plate, for example, the 1,3 and 3,1 modes, each having two parallel nodal lines, could be superimposed to generate nodal patterns that have either two diagonal nodal lines, or an inner, nearly circular one; these are similar to the 1,3 and 3,1 degenerate modes of a square membrane, except that the boundaries are free, not fixed. But, of course, for the actual plate, if $v \neq 0$, the nodal lines of the 1,3 and 3,1 modes are not quite straight or parallel.

Further in this part of his book, Rayleigh went on to consider the *fundamental* mode of the free square plate, which is known to have two straight nodal lines passing through the middle points of opposite sides; i.e., the first doubly antisymmetric mode. He approximated the solution by assuming

$$w(x, y, t) = xy \cos \omega t, \tag{34}$$

in terms of centrally located axes, pointing out that the xy shape is the exact solution for the plate loaded by equal and opposite *static* concentrated forces in the four corners, and that this free vibration approximation should be accurate if the square plates have large, concentrated masses (M) in the four corners. He then used his method ($V_{\max} = T_{\max}$) to calculate the corresponding natural frequency of the system. For $M = 0$ (no corner masses), he gave the result

$$\omega^2 = \frac{24Eh^2}{\rho(1+v)a^4}, \tag{35}$$

saying that "the error is probably not great." One finds out (cf. Ref. [10], Table C15) that the error in this frequency (ω) is 5.3% if $v = 0.3$.

Returning to Rayleigh's 1911 paper [8], in the second paragraph he said, "I wish to call attention to a remarkable memoir by W. Ritz in which, somewhat on the above lines, is developed with great skill what may be regarded as a practically complete solution of the problem of Chladni's figures on square plates." This is certainly a nice compliment to Ritz. But it would have been more proper, and more correct, if "somewhat on the above lines" had been omitted, for there is considerable difference between Ritz's second paper [4] and what Rayleigh showed for plate problems in his book [1]. The only plate problem that Rayleigh actually solved was the one described above for the square plate with corner masses. And this was done simply by setting $V_{\max} = T_{\max}$ (or the equivalent), *without the use of multiple admissible functions and frequency minimization*, which is the essence of Ritz's method.

Rayleigh [8] subsequently went on to say, referring to Ritz's paper: "As it has been said, the general method of approximation is very skillfully applied, but I am surprised that Ritz should have regarded the method itself as new. An integral involving an unknown arbitrary function is to be made a minimum. The unknown function can be represented by a series of known functions with arbitrary coefficients—accurately if the series be continued to infinity, and approximately by a few terms. When the number of coefficients, also called generalized coordinates, is finite, they are of course to be determined by ordinary methods so as to make the integral a minimum. It was in this way that I found the correction for the open end of an organ-pipe using a series with two terms to express the velocity at the mouth. The calculation was further elaborated in *Theory of Sound*, Vol. 2, Appendix A. I had supposed that this treatise abounded in applications of the method in question (see Sections 88–91, 182, 209, 210, 265); but perhaps the most explicit formulation of it is in a more recent paper [18], where it takes almost exactly the shape employed by Ritz. From the title it will be seen that I hardly expected the method to be so successful as Ritz made it in the case of higher modes of vibration."

Because Rayleigh put forth that his hydrodynamics paper [18], was "perhaps the most explicit formulation" of the method, and that "it takes almost exactly the shape employed by Ritz," it behooves us to look at this work first, among those listed by him. He began by writing down the kinetic and potential energies of a linear, two degree-of-freedom system in its most general form

$$T = \tfrac{1}{2}L\dot{q}_1^2 + M\dot{q}_1\dot{q}_2 + \tfrac{1}{2}N\dot{q}_2^2, \tag{36a}$$

$$V = \tfrac{1}{2}Aq_1^2 + Bq_1q_2 + \tfrac{1}{2}Cq_2^2, \tag{36b}$$

where q_1 and q_2 are the generalized coordinates, and M and B are inertial and restoring coupling coefficients, respectively. From this, he said, one obtains (presumably by using Lagrange's Equations, although he did not say) the determinantal equation

$$\begin{vmatrix} A-\omega^2L & B-\omega^2M \\ B-\omega^2M & C-\omega^2N \end{vmatrix} = 0, \tag{37}$$

which yields two frequencies ω_1, ω_2. This then could be a two degree-of-freedom approximation to a linear system, with better approximations possible by using three or more q_i's to yield higher order determinants.

Rayleigh then went on to say [18] that representing a system by less degrees of freedom than it actually has introduces constraints in it, and that the resulting frequencies would all be too high (i.e., upper bounds). In particular, approximation of the fundamental frequency can be improved by adding freedom (i.e., generalized coordinates).

He illustrated this method in Ref. [18] on a problem of liquid sloshing in a rigid, circular cylindrical container, which had been previously discussed by Lamb. The problem is described in Fig. 3. The container is lying on its side, and is half full with a liquid. The liquid has an infinite number of circumferential free vibration modes (i.e., displacements only in the θ-direction); the fundamental one is depicted. The second mode would have two node points on the free surface, in addition to the center ($r = 0$). The displacement of the liquid-free surface completely determines the motion in this mode. Rayleigh expressed it by

$$\eta = -q_2(r/c) + 4q_4(r/c)^3 - 6q_6(r/c)^5 + \cdots, \tag{38}$$

(see Fig. 3). The potential energy in the displaced position is then

$$V = 2\int_0^c \frac{g\eta^2}{2}\,\mathrm{d}r = 4gc\left(\frac{1}{3}q_2^2 - \frac{4}{5}q_2q_4 + \frac{4}{7}q_4^2 + \cdots\right) \tag{39}$$

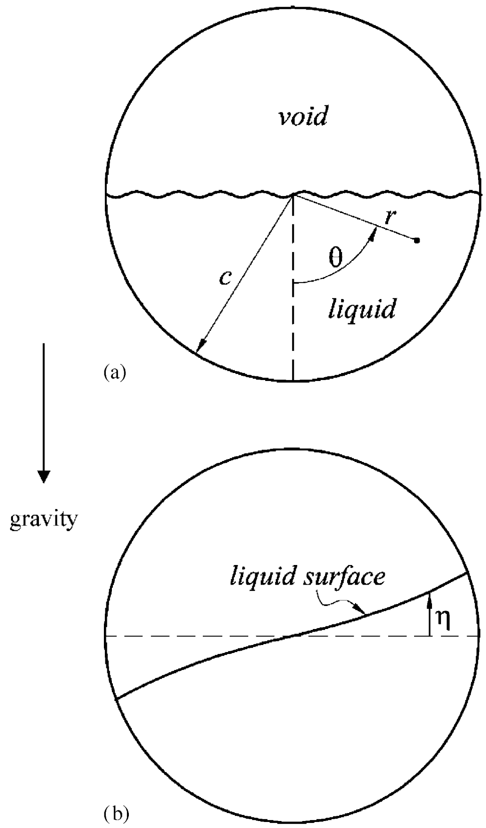

Fig. 3. Liquid sloshing in a half–full cylindrical container: (a) equilibrium, (b) first circumferential sloshing mode.

(assuming unit density, and letting g be the gravitational constant). He then assumed a stream function, as used in classical hydrodynamics (inviscid, incompressible flow), related it to $\dot{\eta}$, and expressed the kinetic energy (T) in terms of the $\dot{q}_i$. As a first approximation, he retained only the first term of Eq. (38). As an improved approximation, he kept the first two terms. This generated T and V in the form of Eqs. (36), from which the frequency determinant (37) could immediately be written and solved for ω_1 and ω_2. The fundamental frequency (ω_1) thus obtained was found to be only 0.4% lower (a closer upper bound to the exact value) than the first approximation. He mentions the possibility of continuing in a similar manner, using the third term in Eq. (38), but does not do it.

How similar is Rayleigh's method, as used for the hydrodynamic sloshing problem, to that laid out by Ritz? In both approaches a set of admissible functions are chosen, as in Eq. (38), and V and T are expressed in terms of the generalized coordinates (q_i). But the subsequent logic is different. As described above, Rayleigh assumes that an infinite degree of freedom system can simply be replaced by a finite dof system, to yield approximate ω. There is no clear mathematical minimization procedure present. But if one looks more closely at Rayleigh's procedure, one realizes that Lagrange's equations, when applied to the energy functionals (36), are minimizing equations. For the two d.o.f. system of Eqs. (36) they yield the *exact* ordinary differential equations in terms of q_1 and q_2 corresponding to this minimum. But Rayleigh did not mention that a mathematical minimizing process is occurring. Rather, he explained in detail how the presence of constraints in the generalized coordinates yields frequencies which are too high, and giving additional freedom to the system improves their accuracy. Ritz, on the other hand, said that the frequencies are too high (or, at least, never less than the exact values) if only a finite number of admissible functions is used in the energy functionals, because the mathematical minimizing procedure would yield a lower minimum if additional terms are taken.

Let us now take up "the correction for the open end of an organ-pipe" mentioned by Rayleigh in his 1911 paper [8], which is supposed to be another example of the Ritz method. This appears as the very last section of his Volume 2 [2]. Because the longitudinal velocity (v) of the air is not constant, a correction is made by assuming that it varies with the radius (r) within the tube as

$$v = 1 + a_2 r^2 + a_4 r^4, \tag{40}$$

where a_2 and a_4 are undetermined coefficients. As explained by Rayleigh, the problem is to minimize the total kinetic energy of the air, including both that within the tube and that outside of the opening, by a lengthy, two-part analysis. This results in a complicated expression for the total kinetic energy, which involves terms with a_2, a_4, a_2^2, a_4^2 and $a_2 a_4$. Rayleigh determines the minimum by intricate algebraic manipulations, instead of taking partial derivatives with respect to a_2 and a_4, which would be the Ritz approach.

Looking at Sections 88 and 89 in Ref. [1], which Rayleigh [8] also described as being equivalent to Ritz's [4] method, one finds a suggested iterative procedure (p. 110) for improving the estimated frequency when using the Rayleigh method, and a lengthy discussion of how mass or constraint added to a system affects its frequencies. In Section 89, one finds the Rayleigh approximate solution (17) for the vibrating string described here earlier. As mentioned earlier, a minimum upper bound frequency for the problem was obtained by varying the exponent n, but only a single admissible function (17) was used.

In Section 90, Rayleigh presented a theoretical study of what happens to a mechanical system if it is changed slightly, so that $T \rightarrow T + \delta T$ and $V \rightarrow V + \delta V$, again showing how frequencies are generally increased or decreased. In Section 91, he applied the equations of Section 90 to a string having variable density. He assumed

$$w(x,t) = \sum_m \phi_m(t) \sin \frac{m\pi x}{l}, \tag{41}$$

for the mode shapes. The method is applied to the string having a small mass added to its midpoint. Approximate frequencies are calculated, but there is no minimization process.

In Section 182, a cantilever beam is deflected by a *static* force at its free end, giving the deflected shape $w = -3lx^2 + x^3$ ($x = l$ is the free end). Using this shape, V and T for the free vibration are calculated, yielding a fundamental frequency (ω) that is 1.4% higher than the exact value. The static force is next applied at an arbitrary distance $x = c$ from the free end. Determining a new w, and calculating V and T, the frequency is determined in terms of c. Rayleigh said that taking $c = 3l/4$ results in a frequency only 1.2% too high. But he did not minimize ω.

The problem of the free vibrations of an *almost circular* membrane was analyzed by Rayleigh in Sections 209 and 210. The boundary radius was taken as $r = a + \delta r$, where a is a constant and δr is a perturbation. The solution to the differential equation in terms of Bessel functions is similarly perturbed to obtain frequencies. In particular, the elliptical membrane having small eccentricity is analyzed. But energy is not mentioned in these sections, so they do not involve the Rayleigh or Ritz methods.

The last example cited by Rayleigh in his 1911 paper [8] as being equivalent to Ritz's method was Section 265 of his classic second volume [2]. This involves the acoustic resonance of air in a closed tube of variable cross-section. He assumes the same sin $\pi x/l$ air displacement along the length (l) of the tube as for the uniform cross-section, calculates the potential (V) and kinetic (T) energies. Setting $V_{\max} = T_{\max}$, he obtains an approximate value (an upper bound) for the fundamental resonant frequency. But this is exactly the method which bears his name, described in Section 2 of this paper. There are no multiple trial functions involved, nor minimization of the frequency, which are present in Ritz's method.

To complete the summary of Rayleigh's 1911 paper [8], where he complained about lack of recognition by Ritz [4], he devoted the second half of it to working out an improved solution to the problem of vibration of a completely free square plate, using ordinary algebraic polynomials as admissible functions. He considered improving upon the fundamental frequency that he had found in *Theory of Sound* [1], taking a single generalized coordinate and setting $V_{\max} = T_{\max}$, by generalizing Eq. (34) to

$$w(x,y,t) = xy[q_1 + q_2(x^2 + y^2) + q_3(x^4 + y^4) + q_4x^2y^2 + \cdots] \cos \omega t. \tag{42}$$

In the same manner with which he had dealt with the liquid sloshing problem [18], he retained the first two terms of Eq. (42), substituted them into the T and V integrals to generate Eqs. (36), and then used Eq. (37) to determine the first two ω. The resulting fundamental frequency was 4.0% less than that of the one-term solution, given as Eq. (34). But, as he admitted "the value thus obtained is not so low, and therefore not so good, as that derived by Ritz."

5. Further discussion

To describe the Ritz method briefly, one could say that it solves a boundary value or eigenvalue problem by assuming a solution in the form of a series of admissible functions (satisfying at least the geometric boundary conditions), each having an arbitrary coefficient, and minimizes the appropriate energy functional directly. Thus, as Ritz and others have described it, it is a *direct* method of solving a variational problem; that is, *not* employing the classical Euler–Lagrange differential equation to first generate equations of motion, which must then be solved.

In the example where Rayleigh used two admissible functions, the liquid sloshing problem [18], he argued on physical grounds that adding terms to the set of admissible functions decreases the unnecessary constraint within the system, and therefore improves the solution. In the organ-pipe problem Rayleigh [2, volume 2, Appendix A] did use two admissible functions, along with an energy approach. But the functional was ultimately minimized by intricate algebraic manipulations, instead of taking partial derivatives as in the Ritz method. For the vibrating string problem ([1, Section 89]), only a single term admissible function (17) was used to obtain the approximate frequency (18) in terms of the exponent (n). The present writer has looked further through Rayleigh's published works, and finds no other example of frequency minimization.

The Ritz method may also be arrived at on physical grounds for a free vibration problem, instead of as a variational problem, in the following manner. Let $T_{\max} = \omega^2 T^*_{\max}$; that is, $T^*_{\max}$ is simply the integral for the maximum kinetic energy of the system during a cycle of motion, with the constant ω^2 factored out. Thus, from Eq. (5)

$$\omega^2 = \frac{V_{\max}}{T^*_{\max}}. \tag{43}$$

The R.H.S of Eq. (43) is called "Rayleigh's Quotient" by many. Using Rayleigh's method, one could substitute the assumed mode shape into the numerator and denominator integrals of Eq. (43) to obtain an approximate ω^2. If the mode shape (i.e., eigenfunction) is given additional freedom, by representing it with a *series* of admissible functions as in, for example, Eq. (25), then the best-possible (i.e., the lowest) frequency may be found directly from the minimizing equations

$$\frac{\partial \omega^2}{\partial a_{ij}} = \frac{\partial}{\partial a_{ij}}\left(\frac{V_{\max}}{T^*_{\max}}\right) = 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n). \tag{44}$$

Differentiating the quotient above yields

$$\frac{\partial}{\partial a_{ij}}(V_{\max} - T_{\max}) = 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n), \tag{45}$$

where $V_{\max} - T_{\max}$ is the standard combination of integrals to be minimized, as Ritz [3] presented it. A similar physical argument may be used to establish buckling loads resulting from the Ritz method. In his lengthy first paper, Ritz [3] did present (p. 58) Eq. (43) as a quotient of integrals to be minimized in a single place, where he discussed applying his method to the vibrating string problem.

Interestingly, in describing one method of writing the minimizing equations (21) for the energy functional (19), Ritz [3] arrived at (*his* Eq. (41) on p. 38) what we now would call a *Galerkin* integral, although Galerkin's expository paper on the subject [19] did not appear until 1915. The transformation of the Ritz minimizing equations to the Galerkin form, in general, is described in detail in the monograph by Kantorovich and Krylov [20] (cf. pp. 263–265 and 272–275). Actually, many Russian analysts call the Galerkin method the "Bubnov–Galerkin" method, to recognize the earlier publication (1913) of Bubnov [21] who showed a different method of arriving at the same algebraic equations as Timoshenko did in applying the *Ritz* method to some elastic stability problems. This different method is equivalent to what Galerkin presented two years later in a more general form. The background of this is discussed well in the monograph by Mikhlin [22] (see pp. 28–31 of his introductory, historical review chapter). But, it is interesting that Ritz showed this approach a few years earlier than either Bubnov or Galerkin. This fact was also recognized by Timoshenko [23, (see his footnote on p. 347)].

Ritz devoted a great deal of his first paper [3] to questions of convergence of series solutions. The last sentence he wrote was therefore particularly important to him: "After the above examples, in application of the new calculation methods also to cases where the convergence proof is yet lacking, the physicist need not feel alarmed by this defect."

6. Concluding remarks

After a lengthy and careful study of all possible relevant works of Rayleigh and Ritz, the present writer concludes that the method of Ritz, as presented by him, is significantly different from what Rayleigh showed. The Rayleigh method, as described above in Section 2, is still a very useful approach for many eigenvalue problems (e.g., free vibration or buckling), but in the words of Mikhlin [22, p. xxi], "the Ritz method is a far-reaching generalization of the so-called 'Rayleigh Method'." That is, while a first approximation to a vibration frequency may be obtained by the Rayleigh method, using a single admissible function for the mode shape in Eq. (43), much better results are typically obtained by using the Ritz method with a series of admissible functions, writing the minimizing Eqs. (45).

As it has been shown, Rayleigh did solve a few problems in ways similar to that used subsequently by Ritz, but *not the same*. That is, as discussed earlier in Sections 4 and 5, Rayleigh solved three problems which involved minimization of a functional, but not by the straightforward, direct Ritz's method followed by scores of subsequent analysts. Therefore, the present writer concludes that Rayleigh's name should not be attached to the Ritz method; that is, the "Rayleigh–Ritz method" is an improper designation.

It is also the opinion of the writer that Lord Rayleigh made the greatest contributions to the study of mechanical vibrations of any person of his time. His *Theory of Sound* [1], as well as other published papers, increased the understanding of vibrational phenomena tremendously more than a century ago. Even now, when read carefully, they still provide excellent insight into the subject.

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